

# CUBIC HAMILTONIANS

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ABSTRACT. We determine a precise necessary and sufficient condition for completeness of the Hamiltonian vector field associated to a homogeneous cubic polynomial on a symplectic plane.

## 0. INTRODUCTION

The flow of the Hamiltonian vector field generated by a smooth function on a symplectic manifold is a familiar object of study. Let the symplectic manifold be simply a symplectic vector space: the Hamiltonian flow generated by a homogeneous linear function is a one-parameter group of translations; the Hamiltonian flow generated by a homogeneous quadratic function is a one-parameter group of linear symplectic transformations. In each of these two cases, the Hamiltonian flow is complete: each maximal integral curve of the Hamiltonian vector field is defined for all time. The case of cubic Hamiltonian functions is different: for some cubics the flow is complete whereas for others it is incomplete.

Our primary objective in this paper is to establish a simple necessary and sufficient condition for the cubic  $\psi$  on a symplectic plane  $(Z, \Omega)$  to generate a complete Hamiltonian flow. In Section 1 we associate with  $\psi$  a suitably symmetric linear map from  $Z$  to the symplectic Lie algebra  $\mathfrak{sp}(Z, \Omega)$ ; following this map with the determinant yields a quadratic map  $\Delta : Z \rightarrow \mathbb{R}$ . In Section 2 we analyze an arbitrary integral curve  $z : I \rightarrow Z$  of the Hamiltonian vector field  $\xi^\psi$  defined by  $\psi$ ; we find that the second time-derivative  $\overset{\circ\circ}{z}$  equals  $2Fz$ , where the scalar function  $F := \Delta \circ z : I \rightarrow \mathbb{R}$  satisfies the equation  $\overset{\circ\circ}{F} = 6F^2$  familiar from the theory of elliptic functions. In Section 3 we achieve our primary objective, proving that the Hamiltonian vector field  $\xi^\psi$  is complete if and only if the determinant  $\Delta$  is identically zero; beyond this, we comment on the nonconstant integral curves of  $\xi^\psi$  in the complete case and the incomplete case. Finally, we assemble several remarks on issues arising from the main body of the paper: in particular, we remark that  $\Delta$  is identically zero if and only if  $\psi$  is a monomial; these remarks we plan to develop more fully in subsequent papers.

In a subsequent paper we also plan to present a similar treatment of quartic Hamiltonian functions; for now, we merely note one difference between the cubic case and the quartic case. In the cubic case, the scalar function  $F$  satisfies the differential equation  $\overset{\circ\circ}{F} = 6F^2$  whose elliptic solutions are always Weierstrass Pe functions associated to triangular lattices, with  $g_2$  zero; in the quartic case, the corresponding scalar functions include Weierstrass functions associated to rectangular lattices, with  $g_2$  nonzero.

## 1. SYMPLECTIC ALGEBRA

Let  $(Z, \Omega)$  be a real symplectic vector space: thus,  $Z$  is a vector space and  $\Omega : Z \times Z \rightarrow \mathbb{R}$  a nonsingular alternating bilinear form. Though it is not necessary for some of what we shall say, we suppose throughout that  $Z$  is two-dimensional, so that  $(Z, \Omega)$  is a symplectic *plane*. The

symplectic algebra  $\text{sp}(Z, \Omega)$  is the (commutator bracket) Lie algebra comprising all linear maps  $C : Z \rightarrow Z$  such that for all  $x, y \in Z$

$$\Omega(Cx, y) + \Omega(x, Cy) = 0.$$

As a vector space,  $\text{sp}(Z, \Omega)$  is canonically isomorphic to the space of all symmetric bilinear forms on  $Z$ : to  $C \in \text{sp}(Z, \Omega)$  there corresponds the symmetric bilinear form

$$Z \times Z \rightarrow \mathbb{R} : (x, y) \mapsto \Omega(x, Cy).$$

Now, let  $\psi : Z \rightarrow \mathbb{R}$  be a homogeneous cubic polynomial. To  $\psi$  we associate the (fully) symmetric trilinear function  $\Psi : Z \times Z \times Z \rightarrow \mathbb{R}$  with value at  $(x, y, z) \in Z \times Z \times Z$  given by

$$\Psi(x, y, z) = \psi(x + y + z) - \{\psi(y + z) + \psi(z + x) + \psi(x + y)\} + \psi(x) + \psi(y) + \psi(z).$$

When  $z \in Z$  is fixed,  $\Psi(x, y, z)$  is symmetric bilinear in  $(x, y) \in Z \times Z$ ; it follows that there exists a unique  $\Gamma_z \in \text{sp}(Z, \Omega)$  such that for all  $x, y \in Z$

$$\Psi(x, y, z) = 2\Omega(x, \Gamma_z y).$$

Full symmetry of  $\Psi$  guarantees that the resulting linear map

$$\Gamma^\psi = \Gamma : Z \rightarrow \text{sp}(Z, \Omega)$$

is symmetric in the sense that for all  $x, y \in Z$

$$\Gamma_x y = \Gamma_y x.$$

Note that if  $z \in Z$  then

$$2\Omega(z, \Gamma_z z) = \Psi(z, z, z) = \{27 - (3 \times 8) + 3\}\psi(z) = 6\psi(z)$$

or

$$\psi(z) = \frac{1}{3}\Omega(z, \Gamma_z z).$$

Differentiation of this formula for  $\psi$  yields the result that if  $v, z \in Z$  then

$$\psi'_z(v) = \frac{1}{3}\{\Omega(v, \Gamma_z z) + \Omega(z, \Gamma_v z) + \Omega(z, \Gamma_z v)\}$$

whence by symmetry of  $\Gamma : Z \rightarrow \text{sp}(Z, \Omega)$  it follows that

$$\psi'_z(v) = \Omega(v, \Gamma_z z).$$

Of course, as  $\psi$  is a cubic, the first derivative  $\psi'_z$  is quadratic in  $z \in Z$ . As a bilinear form, the second derivative  $\psi''_z$  at  $z \in Z$  furnishes another means of introducing  $\Psi$  and  $\Gamma$ : indeed, if also  $x, y \in Z$  then

$$\psi''_z(y, x) = \Psi(x, y, z) = 2\Omega(x, \Gamma_z y).$$

This equation represents  $\psi''_z$  by  $2\Gamma_z$  relative to the symplectic form  $\Omega$ ; consequently, the classical Hessian of  $\psi$  is  $\text{Det}(2\Gamma_z)$ .

According to the Cayley-Hamilton theorem, if  $z \in Z$  then

$$\Gamma_z \Gamma_z - (\text{Tr } \Gamma_z) \Gamma_z + (\text{Det } \Gamma_z) I = 0$$

whence the fact that  $\Gamma_z \in \text{sp}(Z, \Omega)$  is traceless implies that

$$\Gamma_z \Gamma_z = -(\text{Det } \Gamma_z) I.$$

We define the scalar function  $\Delta^\psi = \Delta : Z \rightarrow \mathbb{R}$  by requiring that for each  $z \in Z$

$$\Delta(z) = -(\text{Det } \Gamma_z)$$

so that

$$\Gamma_z \Gamma_z = \Delta(z) I.$$

**Theorem 1.** *If  $z \in Z$  then  $\Delta(\Gamma_z z) = \Delta(z)^2$ .*

*Proof.* If  $z = 0$  then both sides of the alleged equation plainly vanish. If  $z \neq 0$  then apply the special case  $\Gamma_{\Gamma_z z} z = \Gamma_z \Gamma_z z$  of symmetry repeatedly: a first application gives

$$\Delta(\Gamma_z z) z = \Gamma_{\Gamma_z z} \Gamma_{\Gamma_z z} z = \Gamma_{\Gamma_z z} \Gamma_z \Gamma_z z = \Gamma_{\Gamma_z z} \Delta(z) z$$

and a second application gives

$$\Delta(z) \Gamma_{\Gamma_z z} z = \Delta(z) \Gamma_z \Gamma_z z = \Delta(z) \Delta(z) z = \Delta(z)^2 z$$

whence the alleged equation follows by cancellation.  $\square$

## 2. CUBIC HAMILTONIANS

We shall now view  $(Z, \Omega)$  as a symplectic manifold in the natural way. Thus, the vector space  $Z$  is naturally a smooth manifold; if  $z \in Z$  then there is a natural isomorphism from the vector space  $Z$  to the tangent space  $T_z Z$  sending  $v \in Z$  to the directional derivative operator  $v|_z \in T_z Z$  given by the rule that whenever  $f : Z \rightarrow \mathbb{R}$  is a smooth map,

$$v|_z(f) = f'_z(v) = \frac{d}{dt} f(z + tv)|_{t=0}.$$

Also,  $\Omega$  serves double duty as a nonsingular alternating bilinear form on the vector space  $Z$  and as a nonsingular closed two-form on the smooth manifold  $Z$ ; explicitly, if  $x, y, z \in Z$  then the value  $\Omega_z$  of the two-form at  $z$  is given by

$$\Omega_z(x|_z, y|_z) = \Omega(x, y).$$

When  $f : Z \rightarrow \mathbb{R}$  is a smooth (Hamiltonian) function, the corresponding Hamiltonian vector field  $\xi^f \in \text{Vec} Z$  on  $Z$  is defined by the requirement

$$\xi^f \lrcorner \Omega = -df$$

where  $\lrcorner$  signifies contraction as usual. An integral curve of the vector field  $\xi^f$  is a smooth map  $z : I \rightarrow Z$  (on some open interval  $I \ni 0$ ) satisfying the Hamilton equations: for each  $t \in I$  the tangent vector to  $z$  at  $t$  equals the value of  $\xi^f$  at  $z_t$ , thus

$$\dot{z}_t = \xi^f_{z_t}.$$

We shall focus on the case of a homogeneous cubic  $\psi : Z \rightarrow \mathbb{R}$  as Hamiltonian function. The value of  $\xi^\psi$  at  $z \in Z$  is a vector made tangent at  $z$ : say

$$\xi_z^\psi = x^\psi(z)|_z$$

with  $x^\psi : Z \rightarrow Z$  a smooth vector-valued function. Now, let  $v, z \in Z$ : on the one hand,

$$(\xi^f \lrcorner \Omega)_z(v|_z) = \Omega_z(\xi_z^\psi, v|_z) = \Omega_z(x^\psi(z)|_z, v|_z) = \Omega(x^\psi(z), v);$$

on the other hand,

$$-d\psi_z(v|_z) = -\psi'_z(v) = -\Omega(v, \Gamma_z z) = \Omega(\Gamma_z z, v).$$

As the symplectic form  $\Omega$  is nonsingular, it follows that

$$x^\psi(z) = \Gamma_z z.$$

Accordingly, the Hamilton equation for  $z : I \rightarrow Z$  reads

$$\dot{z} = \Gamma_z z.$$

Let  $z : I \rightarrow Z$  be a solution of this Hamilton equation. Take a further derivative: as  $\Gamma$  is symmetric,

$$\dot{\dot{z}} = \Gamma_{\dot{z}} z + \Gamma_z \dot{z} = 2\Gamma_z \dot{z} = 2\Gamma_z \Gamma_z z$$

by a further application of the Hamilton equation. Recall that if  $w \in Z$  then  $\Gamma_w \Gamma_w = \Delta(w)I$  and write

$$F := \Delta \circ z : I \rightarrow \mathbb{R}.$$

It then follows that  $z : I \rightarrow Z$  satisfies the second-order equation

$$\overset{\circ\circ}{z} = 2Fz.$$

Note here that  $\Delta$  is defined on the whole space  $Z$  while  $F$  is defined only along the integral curve  $z$ .

**Theorem 2.** *The scalar function  $F$  satisfies the second-order equation*

$$\overset{\circ\circ}{F} = 6F^2.$$

*Proof.* From the definition

$$FI = \Gamma_z \Gamma_z$$

we deduce by repeated differentiation that

$$\overset{\circ}{F} I = \Gamma_z \Gamma_z + \Gamma_z \Gamma_z^\circ$$

and

$$\overset{\circ\circ}{F} I = \Gamma_z \Gamma_z^\circ + 2\Gamma_z \Gamma_z^\circ + \Gamma_z \Gamma_z^{\circ\circ}.$$

Here, the first and last terms on the right both equal  $2F\Gamma_z \Gamma_z = 2F^2 I$  on account of  $\overset{\circ\circ}{z} = 2Fz$  while  $\Gamma_z \Gamma_z^\circ$  equals  $F^2 I$  on account of  $\overset{\circ}{z} = \Gamma_z z$  and Theorem 1.  $\square$

We may at once deduce a first-order integral of this second-order equation: multiply through by  $2 \overset{\circ}{F}$  to obtain

$$2 \overset{\circ}{F} \overset{\circ\circ}{F} = 12F^2 \overset{\circ}{F}$$

from which there follows

$$(\overset{\circ}{F})^2 = 4F^3 - g_3$$

for some real constant  $g_3$ . This notation is deliberately chosen to accord with the theory of elliptic functions. In fact, the solutions to this first-order differential equation are as follows:

- if  $g_3$  is nonzero then  $F(t) = \wp(t - a)$  for some real  $a$  where  $\wp$  is the Weierstrass Pe function associated to a triangular lattice (the so-called equianharmonic case);
- if  $g_3$  is zero then either  $F(t) = (t - a)^{-2}$  for some real  $a$  or  $F$  is identically zero.

Note that when  $F$  is a (shifted) Weierstrass Pe function,  $\overset{\circ\circ}{z} = 2Fz$  is a (vectorial) Lamé equation and may be solved accordingly; for example, see page 285 of [Forsyth].

### 3. COMPLETENESS CHARACTERIZED

We continue to let  $\Gamma : Z \rightarrow \text{sp}(Z, \Omega)$  be the symmetric linear map corresponding to the homogeneous cubic  $\psi : Z \rightarrow \mathbb{R}$  on the symplectic plane  $(Z, \Omega)$ ; we also continue to let  $z : I \rightarrow Z$  be an integral curve of the associated Hamiltonian vector field  $\xi^\psi$ . We shall suppose that the curve  $z$  has initial point  $z_0$  and hence initial velocity  $\overset{\circ}{z}_0 = \Gamma_{z_0} z_0$ . Our aim in this section is to decide precisely when such an integral curve may be defined for all time; that is, precisely when the maximal domain of definition  $I$  is  $\mathbb{R}$  itself.

The critical case is decided immediately. Let  $\xi^\psi$  (equivalently,  $d\psi$ ) vanish at  $z_0$ ; thus,  $z$  has initial velocity  $\overset{\circ}{z}_0 = \Gamma_{z_0} z_0 = 0$ . In this critical case, the solution  $z : I \rightarrow Z$  is plainly given by  $z_t = z_0$  for all  $t \in I$  and the maximal  $I$  is indeed  $\mathbb{R}$ . In this connexion, note further that if an integral curve  $z : I \rightarrow Z$  vanishes at any point then so does its velocity vector and hence  $z$  itself is identically zero.

Now let the integral curve  $z : I \rightarrow Z$  be other than critical: thus,  $\Gamma_{z_0} z_0 = \overset{\circ}{z}_0 \neq 0$  and of course  $z_0 \neq 0$ . We distinguish two cases.

For the first case, suppose there exists some  $s \in I$  such that  $0 \neq F(s) = \Delta(z_s)$  and therefore  $\overset{\circ}{F}(s) = \overset{\circ}{F}(s)^2 > 0$ . The comments after Theorem 2 show that  $F$  has a double pole at some real  $a$ ; thus  $\Gamma_{z_t}\Gamma_{z_t} = F(t)I$  is unbounded as  $t \rightarrow a$  and so  $z_t$  itself is unbounded as  $t \rightarrow a$ . In this case, the maximal domain of  $z$  omits  $a$  and thereby falls short of  $\mathbb{R}$ .

For the second case, suppose that  $F(t) = 0$  whenever  $t \in I$ . Note that the linear map  $\Gamma_{z_0}$  kills  $\Gamma_{z_0}z_0$  (because  $\Gamma_{z_0}\Gamma_{z_0} = F(0)I = 0$ ) but does not kill  $z_0$  (because  $\Gamma_{z_0}z_0 = \overset{\circ}{z}_0 \neq 0$ ); thus  $z_0$  and  $\overset{\circ}{z}_0$  constitute a basis for the plane  $Z$  and so

$$\{s(z_0 + t \overset{\circ}{z}_0) : s, t \in \mathbb{R}\} = (Z \setminus \mathbb{R} \overset{\circ}{z}_0) \cup \{0\}.$$

The supposition  $F \equiv 0$  implies that  $\overset{\circ}{z} = 2Fz \equiv 0$  so that  $z_t = z_0 + t \overset{\circ}{z}_0$  for all  $t \in I$ ; essentially as in the critical case, the maximal  $I$  is therefore  $\mathbb{R}$ . Now  $\Delta$  vanishes on  $z_0 + t \overset{\circ}{z}_0$  whenever  $t \in \mathbb{R}$  (as  $F$  is identically zero) and hence vanishes on  $s(z_0 + t \overset{\circ}{z}_0)$  whenever  $s, t \in \mathbb{R}$  (as  $\Delta$  is homogeneous); the continuous function  $\Delta$  now vanishes on the dense set  $(Z \setminus \mathbb{R} \overset{\circ}{z}_0) \cup \{0\}$  and therefore vanishes on the whole of  $Z$ . This proves that if  $\Delta$  vanishes on the image of some non-critical integral curve then  $\Delta$  vanishes identically.

We may now marshal these facts towards the following result.

**Theorem 3.** *Let  $\psi : Z \rightarrow \mathbb{R}$  be a homogeneous cubic and  $\Delta^\psi$  the associated determinant.*

- *If  $\Delta^\psi \equiv 0$  then  $\xi^\psi$  is complete; each non-constant integral curve is an affine line.*
- *If  $\Delta^\psi \neq 0$  then  $\xi^\psi$  is incomplete; only the constant integral curves are defined for all time.*

*Proof.* If  $\Delta \equiv 0$  then each maximal integral curve  $z$  has  $F \equiv 0$  so that  $\overset{\circ}{z} = 2Fz \equiv 0$  and  $z$  on  $\mathbb{R}$  is affine, as we have seen. If  $\Delta \neq 0$  and the integral curve  $z$  is not critical, then  $F \neq 0$  so that  $z$  experiences finite-time blow-up, as we have seen.  $\square$

Looking ahead to the next section, we remark that  $\Delta^\psi$  is identically zero if and only if  $\psi$  is monomial in the sense that there exists  $w \in Z$  such that for all  $z \in Z$

$$\psi(z) = \frac{1}{3}\Omega(w, z)^3.$$

#### 4. REMARKS

In this closing section, we record a number of miscellaneous remarks that stem from the body of this paper.

##### COORDINATE EXPRESSIONS

Though our whole approach has been intentionally coordinate-free, it is also of interest to see the development in terms of linear symplectic coordinates, not least because this may offer glimpses of a fresh perspective on classical invariant theory.

To this end, let  $u, v \in Z$  satisfy  $\Omega(u, v) = 1$  and so constitute a symplectic basis for  $(Z, \Omega)$ . Decompose  $z \in Z$  as

$$z = pu + qv$$

with

$$p = p(z) = \Omega(z, v), \quad q = q(z) = \Omega(u, z).$$

Write

$$\begin{aligned} a &= \Omega(u, \Gamma_u u), & b &= \Omega(u, \Gamma_v u), \\ c &= \Omega(v, \Gamma_u v), & d &= \Omega(v, \Gamma_v v). \end{aligned}$$

With these conventions, the cubic

$$\psi(z) = \frac{1}{3}\Omega(z, \Gamma_z z)$$

has coordinate form

$$\psi(z) = \frac{1}{3}\{ap^3 + 3bp^2q + 3cpq^2 + dq^3\}$$

and the (vector) Hamilton equation

$$\overset{\circ}{z} = \Gamma_z z$$

becomes the familiar scalar pair

$$\overset{\circ}{p} = -\frac{\partial\psi}{\partial q}, \quad \overset{\circ}{q} = \frac{\partial\psi}{\partial p}.$$

The associated determinant

$$\Delta(z) = -(\text{Det } \Gamma_z)$$

assumes the form

$$\Delta(z) = (b^2 - ac)p^2 + (bc - ad)pq + (c^2 - bd)q^2$$

and is the Hessian of  $\psi$  (up to scale). We are not the first to observe that the discriminant

$$(bc - ad)^2 - 4(b^2 - ac)(c^2 - bd)$$

of this quadratic is precisely the discriminant

$$a^2d^2 - 3b^2c^2 + 4ac^3 + 4b^3d - 6abcd$$

of the cubic

$$ap^3 + 3bp^2q + 3cpq^2 + dq^3;$$

for example, see page 60 of [Salmon].

Of course, a purely coordinate-based approach is possible. Let us indicate partial derivatives more succinctly by means of subscripts. With the cubic

$$\psi(z) = \frac{1}{3}\{ap^3 + 3bp^2q + 3cpq^2 + dq^3\}$$

as above, direct computation reveals that  $\psi_{pq}\psi_q - \psi_p\psi_{qq}$  is divisible by  $p$  and  $\psi_{qp}\psi_p - \psi_q\psi_{pp}$  is divisible by  $q$ ; in each case, the quotient is precisely  $2\{(b^2 - ac)p^2 + (bc - ad)pq + (c^2 - bd)q^2\}$  and we recover (twice) the determinant  $\Delta$  in coordinate form. In fact, when the Hamilton equations

$$\overset{\circ}{p} = -\psi_q, \quad \overset{\circ}{q} = \psi_p$$

are differentiated by time once more, they yield precisely

$$\overset{\circ\circ}{p} = \psi_{pq}\psi_q - \psi_p\psi_{qq}, \quad \overset{\circ\circ}{q} = \psi_{qp}\psi_p - \psi_q\psi_{pp}$$

and we recover the scalar components of  $\overset{\circ\circ}{z} = 2Fz$ .

#### CANONICAL FORMS

The simplest type of homogeneous cubic is a monomial: for  $w \in Z$  define  $\psi^w : Z \rightarrow \mathbb{R}$  by requiring that for all  $z \in Z$

$$\psi^w(z) = \frac{1}{3}\Omega(w, z)^3.$$

For this cubic, the corresponding symmetric linear map  $\Gamma^w : Z \rightarrow \text{sp}(Z, \Omega)$  is given by

$$\Gamma_z^w v = \Omega(z, w)\Omega(w, v)w$$

whenever  $z, v \in Z$ , and the associated determinant  $\Delta^w$  is identically zero.

Conversely, let the cubic  $\psi$  with corresponding symmetric linear map  $\Gamma$  be such that the associated determinant  $\Delta$  is identically zero. We claim that  $\psi = \psi^w$  for a unique  $w \in Z$ ; to justify this claim, we may of course assume that  $\Gamma$  is not itself identically zero. Note that if  $z \in Z$  then  $\Gamma_z \Gamma_z = 0$  so that  $\text{Ran } \Gamma_z \subseteq \text{Ker } \Gamma_z$  with equality precisely when  $\Gamma_z \neq 0$ . Note also that if  $x, y \in Z$  then

$$\Gamma_x \Gamma_y + \Gamma_y \Gamma_x = \{\Delta(x+y) - \Delta(x) - \Delta(y)\}I = 0.$$

When  $x, y, z \in Z$  let us write

$$\gamma(x, y, z) = \Gamma_x \Gamma_y z.$$

Observe that this expression is now antisymmetric in its first pair of variables and was already symmetric in its last pair; thus

$$\gamma(x, y, z) = \gamma(x, z, y) = -\gamma(z, x, y) = -\gamma(z, y, x) = \gamma(y, z, x) = \gamma(y, x, z) = -\gamma(x, y, z)$$

and so  $\gamma$  vanishes identically. This proves that if  $x, y \in Z$  then

$$\text{Ran } \Gamma_y \subseteq \text{Ker } \Gamma_x$$

and choosing any  $z \in Z$  with  $\Gamma_z \neq 0$  then gives

$$\text{Ran } \Gamma_z \subseteq \cup_{y \in Z} \text{Ran } \Gamma_y \subseteq \cap_{x \in Z} \text{Ker } \Gamma_x \subseteq \text{Ker } \Gamma_z$$

with equality of the end terms and hence equality throughout, whence

$$\cup_{y \in Z} \text{Ran } \Gamma_y = \cap_{x \in Z} \text{Ker } \Gamma_x$$

is a distinguished line in the plane  $Z$ . Let  $w \in Z$  be a basis vector for this line. If  $z \in Z$  then  $\Gamma_z = \lambda_z(\cdot)w$  for some linearly  $z$ -dependent  $\lambda_z$  in the dual  $Z^*$ : as  $\Gamma_z$  kills  $w$  so does  $\lambda_z$  and therefore  $\lambda_z = \mu_z \Omega(w, \cdot)$  for some  $\mu_z \in \mathbb{R}$  also linear in  $z$ ; this shows that

$$\Gamma_z = \mu_z \Omega(w, \cdot)w$$

for some  $\mu \in Z^*$ . Symmetry of  $\Gamma$  forces  $\mu$  to kill  $w$  so that  $\mu = \nu \Omega(\cdot, w)$  for some  $\nu \in \mathbb{R}$ . In the resulting formula

$$\Gamma_z = \nu \Omega(z, w) \Omega(w, \cdot)w$$

the cube root of the scalar  $\nu$  may be absorbed into  $w$ ; this renders  $w$  unique and we conclude that  $\Gamma = \Gamma^w$  as claimed.

Thus, the assignment  $w \mapsto \Gamma^w$  is a (cubic!) bijection from  $Z$  to the set of all symmetric linear maps  $Z \rightarrow \text{sp}(Z, \Omega)$  for which the associated determinant  $\Delta$  is identically zero.

The same conclusion may be reached efficiently (though prosaically) using coordinates. From the identical vanishing of  $\Delta$  in the form

$$(b^2 - ac)p^2 + (bc - ad)pq + (c^2 - bd)q^2 \equiv 0$$

we deduce (by setting  $q = 0, p = 0$ , and  $pq \neq 0$  in turn) that  $b^2 = ac, c^2 = bd$ , and  $ac = bd$ . Let  $\lambda$  be the cube root of  $a$  and  $\mu$  the cube root of  $d$ : then

$$(\lambda^2 \mu)^3 = a^2 d = a \cdot ad = a \cdot bc = b \cdot ac = b \cdot b^2 = b^3$$

so that  $\lambda^2 \mu = b$  and  $\lambda \mu^2 = c$  likewise; it follows that the cubic is a monomial, namely

$$ap^3 + 3bp^2q + 3cpq^2 + dq^3 = (\lambda p + \mu q)^3.$$

When the determinant  $\Delta$  is not identically zero, there are three possibilities:

- $\Delta(z) = 0$  for  $z$  on a line-pair through 0 and  $\Delta$  takes values of each sign elsewhere;
- $\Delta(z) = 0$  for  $z$  on a line through 0 and  $\Delta$  is positive elsewhere;
- $\Delta(0) = 0$  and  $\Delta$  is positive elsewhere;

and canonical forms may be developed for each of these. In connexion with these possibilities, we remark (from Theorem 1) that if  $\Delta$  takes negative values then it also takes positive values.

EVALUATION OF  $g_3$ 

Let  $\psi : Z \rightarrow \mathbb{R}$  be a homogeneous cubic and let the Hamiltonian vector field  $\xi^\psi$  have  $z : I \rightarrow Z$  as an integral curve. As we have seen,  $\overset{\circ}{z} = 2Fz$  where the scalar function  $F : I \rightarrow \mathbb{R}$  satisfies  $(\overset{\circ}{F})^2 = 4F^3 - g_3$  for some constant  $g_3$  that depends on the integral curve  $z$ .

Let the initial point  $z_0$  be such that  $\psi(z_0) = 0$ ; as the Hamiltonian  $\psi$  is constant along the integral curve, it follows that  $\psi(z_t) = 0$  for all  $t \in I$ . If  $z_0$  itself is zero, then of course  $F \equiv 0$  and  $g_3 = 0$ . Now assume that  $z_0$  is nonzero, so that  $z_t$  is nonzero for all  $t \in I$ . For each  $t \in I$  we have  $0 = 3\psi(z_t) = \Omega(z_t, \overset{\circ}{z}_t)$  whence (as  $Z$  is a plane)  $\overset{\circ}{z}_t$  is parallel to  $z_t$ ; say  $\overset{\circ}{z} = \lambda z$  for some scalar function  $\lambda : I \rightarrow \mathbb{R}$ . On the one hand,

$$Fz = \Gamma_z \Gamma_z z = \Gamma_z \overset{\circ}{z} = \Gamma_z \lambda z = \lambda \Gamma_z z = \lambda \overset{\circ}{z} = \lambda^2 z;$$

on the other hand,

$$2Fz = \overset{\circ}{z} = \overset{\circ}{\lambda} z + \lambda \overset{\circ}{z} = \overset{\circ}{\lambda} z + \lambda^2 z = \overset{\circ}{\lambda} z + Fz.$$

Thus

$$\overset{\circ}{\lambda} = F = \lambda^2$$

and so

$$\overset{\circ}{F} = (\lambda^2)^\circ = 2\lambda \overset{\circ}{\lambda} = 2\lambda F = 2\lambda^3.$$

It follows that in this case,

$$g_3 = 4F^3 - (\overset{\circ}{F})^2 = 4(\lambda^2)^3 - (2\lambda^3)^2 = 0.$$

In short, an initial point  $z_0$  with  $\psi(z_0) = 0$  spawns an integral curve for which  $g_3 = 0$ .

Let us offer some sample computations in coordinates. If  $\psi = \frac{1}{3}(p^3 - q^3)$  then  $\overset{\circ}{p} = -\psi_q = q^2$  and  $\overset{\circ}{q} = \psi_p = p^2$  so that  $\overset{\circ}{\overset{\circ}{p}} = 2(pq)p$  and  $\overset{\circ}{\overset{\circ}{q}} = 2(pq)q$ ; thus  $F = pq$  so  $\overset{\circ}{F} = F_q \psi_p - F_p \psi_q = p^3 + q^3$  and  $g_3 = 4F^3 - (\overset{\circ}{F})^2 = 4p^3 q^3 - (p^3 + q^3)^2 = -(p^3 - q^3)^2$  or  $g_3 = -9\psi^2 \leq 0$ . Similarly, if  $\psi = p^2 q + pq^2$  then  $F = p^2 + pq + q^2$  and  $\overset{\circ}{F} = (q - p)(2p + q)(p + 2q)$ ; after considerable simplification,  $g_3 = 4F^3 - (\overset{\circ}{F})^2$  yields  $g_3 = 27\psi^2 \geq 0$ .

Finally, we remark (without proof - but see page 100 of [Salmon]) that classical invariant theory reappears in general: if

$$\delta = a^2 d^2 - 3b^2 c^2 + 4ac^3 + 4b^3 d - 6abcd$$

denotes the discriminant of the cubic  $3\psi$  then

$$g_3 = -9\delta \psi^2$$

so

$$(\overset{\circ}{F})^2 = 4F^3 + 9\delta \psi^2.$$

## REFERENCES

[Forsyth] A.R. Forsyth, *Theory of Functions of a Complex Variable*, Cambridge, First Edition (1893).

[Salmon] G. Salmon, *Lessons Introductory to the Modern Higher Algebra*, Dublin, First Edition (1859).